AP STATISTICS
TOPIC 3: SETS AND FUNCTIONS

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1. Sets and Elements

A *set* is a collection of *elements*. The elements of a set are sometimes called *members* or *points*. We assume that we can distinguish between different elements, and that we can determine whether or not a given element is in a given set.

The relationship of two elements a and b being the same is *equality* and is denoted $a = b$. The negation of this relation is denoted $a \neq b$, that is, $a \neq b$ means that it is not the case that $a = b$.

The relationship of an element a being a member of a set A is *containment* and is denoted $a \in A$. The negation of this relation is denoted $b \notin A$, that is, $b \notin A$ means that it is not the case that $b \in A$.

A set is determined by the elements it contains. That is, two sets are considered equal if and only if they contain the same elements. We use the symbols "$\Rightarrow$" to mean "implies", and "$\Leftrightarrow$" to mean "if and only if". Then

$$A = B \quad \Leftrightarrow \quad (a \in A \Leftrightarrow a \in B);$$

in English, "A equals B if and only if (a is in A if and only if b is in B)".

Thus we should not think of a set as a "container", but rather as the things being contained. For example, consider a glass of water, and the set of water molecules in the glass. If we pour all of the water into an empty bowl, the bowl now contains the same set of water molecules.

One way of describing a set is by explicitly listing its members. Such lists are surrounded by braces, e.g., the set of the first five prime integers is $\{2, 3, 5, 7, 11\}$. If the pattern is clear, we may use dots; for example, to label the set of all prime numbers as P, we may write $P = \{2, 3, 5, 7, 11, 13, \dots\}$. Thus $2 \in P$ and $23 \in P$, but $1 \notin P$ and $21 \notin P$. As another example, if we denote the set of all integers by $\mathbb{Z}$, we may write $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$. Note that the order of elements in a list is irrelevant in determining a set, for example, $\{5, 3, 7, 11, 2\} = \{2, 3, 5, 7, 11\}$. Also, there is no such thing as the "multiplicity" of an element in a set, for example $\{1, 3, 2, 2, 1\} = \{1, 2, 3\}$.

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2. Subsets

If A and B are sets and all of the elements in A are also contained in B, we say that A is a *subset* of B or that A is *contained* in B and write $A \subset B$:

$$A \subset B \quad \Leftrightarrow \quad (a \in A \Rightarrow a \in B);$$

in English, "A is contained in B if and only (a is in A implies a is in B)".

For example, $\{1,3,5\} \subset \{1,2,3,4,5\}$. Note that any set is a subset of itself. We say that A is a *proper subset* of B is $A \subset B$ but $A \neq B$.

It follows immediately from the definition of subset that

$$A = B \quad \Leftrightarrow \quad (A \subset B \text{ and } B \subset A);$$

in English, "A equals B if and only if (A is a subset of B and B is a subset of A)."

Thus to show that two sets are equal, it suffices to show that each is contained in the other.

A set containing no elements is called the *empty set* and is denoted $\varnothing$. Since a set is determined by its elements, there is only one empty set. Note that the empty set is a subset of any set.

3. Set Operations

We may construct new sets as subsets of existing sets by specifying properties. Specifically, we may have a proposition $p(x)$ which is true for some elements x in a set X and not true for others. Then we may construct the set

$$\{x \in X \mid p(x) \text{ is true}\};$$

this is read "the set of x in X such that $p(x)$". The construction of this set is called *specification.* For example, if we let $\mathbb{Z}$ be the set of integers, the set P of all prime numbers could be specified as $P = \{n \in \mathbb{Z} \mid n \text{ is prime}\}$.

Let A and B be subsets of some "universal set" U and define the following set operations:

$$\begin{aligned} \text{Union:} \quad & A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\} \\ \text{Intersection:} \quad & A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\} \\ \text{Complement:} \quad & A \setminus B = \{x \in U \mid x \in A \text{ and } x \notin B\} \end{aligned}$$

The pictures which correspond to these operations are called *Venn diagrams.*

Example 1. Let $A = \{1,3,5,7,9\}$, $B = \{1,2,3,4,5\}$. Then $A \cap B = \{1,3,5\}$, $A \cup B = \{1,2,3,4,5,7,9\}$, $A \setminus B = \{7,9\}$, and $B \setminus A = \{2,4\}$. $\square$

Example 2. Let A and B be two distinct nonparallel lines in a plane. We may consider A and B as sets of points. Their intersection is a set containing a single point, their union is a set consisting of all points on crossing lines, and the complement of A with respect to B is A minus the point of intersection. $\square$

If $A \cap B = \varnothing$, we say that A and B are *disjoint.*

Example 3. A *sphere* is the set of points in space equidistant from a given point, called its *center*; the common distance to the center is called that *radius* of the sphere. Thus a sphere is the surface of a solid ball.

Take two points in space such that the distance between them is 10, and imagine two spheres centered at these points. Let one of the spheres have radius 5. If the radius of the other sphere is less than 5 or greater than 15, then the spheres are disjoint. If the radius of the other sphere is exactly 5 or 15, the intersection is a single point. If the radius of the other sphere is between 5 and 15, the spheres intersect in a circle. $\square$

The following properties are sometimes useful.

- $A = A \cup A = A \cap A$
- $\varnothing \cap A = \varnothing$
- $\varnothing \cup A = A$
- $A \subset B \Leftrightarrow A \cap B = A$
- $A \subset B \Leftrightarrow A \cup B = B$

The following properties state that union and intersection are commutative and associative operations, and that they distribute over each other. These properties are intuitively clear via Venn diagrams.

- $A \cap B = B \cap A$
- $A \cup B = B \cup A$
- $(A \cap B) \cap C = A \cap (B \cap C)$
- $(A \cup B) \cup C = A \cup (B \cup C)$
- $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
- $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

Since $(A \cap B) \cap C = A \cap (B \cap C)$, parentheses are useless and we write $A \cap B \cap C$. This extends to four sets, five sets, and so on. Similar remarks apply to unions.

The following properties of complement are known as *DeMorgan's Laws*. You should draw Venn diagrams of these situations to convince yourself that these properties are true.

- $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$
- $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

Here are a few more properties of complement:

- $A \subset B \Rightarrow A \cup (B \setminus A) = B$;
- $A \subset B \Rightarrow A \cap (B \setminus A) = \varnothing$;
- $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap B \cap C)$;
- $(A \setminus B) \setminus C = A \setminus (B \cup C)$.

4. Cartesian Product

Let a and b be elements. The *ordered pair* of a and b is denoted (a, b) and is defined as

$$(a, b) = \{\{a\}, \{a, b\}\}.$$

This is the technical definition; think about how it relates to the intuitive approach below.

Intuitively, if a and b are elements, the *ordered pair* with first coordinate a and second coordinate b is something like a set containing a and b, but in such a way that the order matters. We denote this ordered pair by (a, b) and declare that it has the following "defining property":

$$(a, b) = (c, d) \quad \Leftrightarrow \quad (a = c \text{ and } b = d).$$

The *cartesian product* of the sets A and B is denoted $A \times B$ and is defined to be the set of all ordered pairs whose first coordinate is in A and whose second coordinate is in B:

$$A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

Example 4. Let $A = \{1, 3, 5\}$ and let $B = \{1, 4\}$. Then

$$A \times B = \{(1, 1), (1, 4), (3, 1), (3, 4), (5, 1), (5, 4)\}.$$

In particular, this set contains 6 elements. □

In general, if A contains m elements and B contains n elements, where m and n are natural numbers, then $A \times B$ contains mn elements. Consider the case where $A = B$; then $A \times A$ contains m^2 elements. We sometimes write A^2 to mean $A \times A$.

We have the following properties:

- $(A \cup B) \times C = (A \times C) \cup (B \times C)$;
- $(A \cap B) \times C = (A \times C) \cap (B \times C)$;
- $A \times (B \cup C) = (A \times B) \cup (A \times C)$;
- $A \times (B \cap C) = (A \times B) \cap (A \times C)$;
- $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$.

The idea of cartesian product can be extended to k sets. Thus if $A_1, \dots, A_k$ are sets, and *ordered k-tuple* from them is an a list $(a_1, \dots, a_k)$, where $a_1 \in A_1, a_2 \in A_2, \dots, a_k \in A_k$. The cartesian product of these sets is the set of all k-tuples from them:

$$\bigtimes_{i=1}^{k} A_i = \{(a_1, \dots, a_k) \mid a_i \in A_i \text{ for } i = 1, \dots, k\}.$$

In this case,

$$|\bigtimes_{i=1}^{k} a_i| = \Pi_{i=1}^{k} |A_i|.$$

In the case of taking the cartesian product of a set A with itself k times, the product is denoted A^k. We may view A^k as the set of ordered sequences of elements from A of length k:

$$A^k = \{(a_1, \dots, a_k) \mid a_i \in X\}.$$

5. Numbers

The following familiar sets of numbers have standard names:

Natural Numbers:	$\mathbb{N} = \{0, 1, 2, 3, \dots\}$
Integers:	$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
Rational Numbers:	$\mathbb{Q} = \{\frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0\}$
Real Numbers:	$\mathbb{R} = \{$ numbers given by decimal expansions $\}$
Complex Numbers:	$\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R}$ and $i^2 = -1\}$

We have $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.
Standard notation gives subsets of the real numbers, called *intervals*:

- $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$ (closed)
- $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$ (open)
- $[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$
- $(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$
- $(-\infty, b] = \{x \in \mathbb{R} \mid x \leq b\}$ (closed)
- $(-\infty, b) = \{x \in \mathbb{R} \mid x < b\}$ (open)
- $[a, \infty) = \{x \in \mathbb{R} \mid a \leq x\}$ (closed)
- $(a, \infty) = \{x \in \mathbb{R} \mid a < x\}$ (open)

Example 5. Let $A = [1, 5]$ be the closed interval of real numbers between 1 and 5 and let $B = (10, 16)$ be the open interval of real numbers between 10 and 16. Let $C = A \cup B$. Let $\mathbb{N}$ be the set of natural numbers. How many elements are in $C \cap \mathbb{N}$?

Solution. We know that

$$C \cap \mathbb{N} = (A \cup B) \cap \mathbb{N} = (A \cap \mathbb{N}) \cup (B \cap \mathbb{N}).$$

Now $A \cap \mathbb{N}$ is the set of natural numbers between 1 and 5, inclusive, so $A \cap \mathbb{N} = \{1, 2, 3, 4, 5\}$. Also, $B \cap \mathbb{N}$ is the set of natural number between 10 and 16, exclusive, so $B \cap \mathbb{N} = \{11, 12, 13, 14, 15\}$. Now $C \cap \mathbb{N}$ is the union of these set, so $C \cap \mathbb{N} = \{1, 2, 3, 4, 5, 11, 12, 13, 14, 15\}$. Therefore $C \cap \mathbb{N}$ has 10 elements. $\square$

Example 6. Let $A = [1, 4]$ and $B = [3, 8)$ be intervals of real numbers. We will see how to view $A \times B$ as a square in the cartesian plane $\mathbb{R}^2$ which has a boundary on three sides but not of the fourth. How many elements are in $(A \times B) \cap (\mathbb{Z} \times \mathbb{Z})$?

Solution. By a previously stated property of cartesian product, we have

$$(A \times B) \cap (\mathbb{Z} \times \mathbb{Z}) = (A \cap \mathbb{Z}) \times (B \cap \mathbb{Z}).$$

Now $A \times \mathbb{Z} = \{1, 2, 3, 4\}$ and $B \times \mathbb{Z} = \{3, 4, 5, 6, 7\}$. Thus $(A \times B) \cap (\mathbb{Z} \times \mathbb{Z})$ has $4 \cdot 5 = 20$ elements. $\square$

Warning 1. The notation for ordered pair (a, b) is the same as the standard notation for open interval of real numbers, but its meaning is entirely different. This is standard, and you must decide from the context which meaning is intended.

6. Cardinality

The *cardinality* of a set is the number of elements in it. A set is *finite* if the exists a natural number $n \in \mathbb{N}$ such that the cardinality of the set is n. Otherwise, it is infinite. It should be noted that there are *levels of infinity*; that is, there are infinite sets such that one has greater cardinality than the other. We will not investigate this further here.

Set operations relate to cardinality in various ways. Contemplate why each of the following is true.

- $|A \cup B| \leq |A| + |B|$
- $|A \cap B| \leq \min\{|A|, |B|\}$
- $|A \setminus B| \leq |A|$
- $|A \cup B| = |A| + |B| - |A \cap B|$

The *inclusion-exclusion principle* states the cardinality of a union of sets. The first two are given.

- $|A \cup B| = |A| + |B| - |A \cap B|$
- $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

7. Collections

7.1. **Collections.** A *collection* is a set whose elements are themselves sets.

The *power set* of a set X is the collection of all subsets of a given set:

$$\mathcal{P}(X) = \{ \text{ sets } A \mid A \subset X\}.$$

Let X be finite set containing n elements, so that $|X| = n$.

A subset of X may be viewed as a choice, for each element of X, as to whether or not the the element is in the subset. Viewed in this way, we make 2^n choices to determine a subset of X, so

$$|\mathcal{P}(X)| = 2^n.$$

Let $\mathcal{P}_k(X)$ denote the set of all subsets of X which contain k elements:

$$\mathcal{P}_k(X) = \{A \in \mathcal{P}(X) \mid |A| = k\}.$$

There are n choose k ways to select the elements of a member of $\mathcal{P}_k(X)$, so

$$|\mathcal{P}_k(X)| = \binom{n}{k}.$$

7.2. **Partitions.** Let X be a nonempty set. A *partition* of X is a collection of nonempty subsets of X, known as *blocks*, such that every element of X is in exactly one block. Thus, $\mathcal{C} \subset \mathcal{P}(X)$ is a partition of X if both these conditions hold:

(1) $\bigcup_{C \in \mathcal{C}} C = X$;
(2) $C_1, C_2 \in \mathcal{C}$ and $C_1 \neq C_2 \Rightarrow C_1 \cap C_2 = \varnothing$.

Thus a partition covers X with nonoverlapping blocks.

If $\mathcal{C}$ is a partition of X, then

$$|X| = \sum_{C \in \mathcal{C}} |C|.$$

Note that the sets $\mathcal{P}_k(X)$, for $k = 0, \ldots, n$, form a partition of $\mathcal{P}(X)$. Thus

$$2^n = \sum_{k=0}^{n} \binom{n}{k}.$$

8. Functions

8.1. **Functions.** A *function* from a set A to a set B is an assignment of each element of A to a unique element of B. The notation $f : A \to B$ means "f is a function from A to B".

Let $f : A \to B$. If $a \in A$, the unique element of B to which a is assigned is denoted $f(a)$.

We call A the *domain* of f, and we call B the *codomain* of f. The domain of f is often denoted $\mathrm{dom}(f)$. The *range* of f is the subset of B given by

$$\mathrm{range}(f) = \{b \in B \mid b = f(a) \text{ for some } a \in A\}.$$

The *graph* of f is the subset of $A \times B$ given by

$$\mathrm{graph}(f) = \{(a, b) \in A \times B \mid b = f(a)\}.$$

8.2. **Images and Preimages.** If $C \subset A$, the *image of C under f* is the subset of the codomain B which consists of all the elements of B to which f assigns some element from C:

$$f(C) = \{b \in B \mid b = f(c) \text{ for some } c \in C\}.$$

If $D \subset B$, the *preimage of D under f* is the subset of the domain A which consists of all the elements of A which are assigned by f to an element in D:

$$f^{-1}(D) = \{a \in A \mid f(a) \in D\}.$$

8.3. **Injections and Surjections.** We say that f is *injective* (or *one-to-one*) if

$$f(a_1) = f(a_2) \quad \Rightarrow \quad a_1 = a_2,$$

where $a_1, a_2 \in A$.

We say that f is *surjective* (or *onto*) if

$$\text{for every } b \in B \text{ there exists } a \in A \text{ such that } f(a) = b.$$

We say that f is *bijective* if f is injective and surjective.

If A is a finite set, a function $f : A \to A$ is injective if and only if it is surjective.

Two sets have the same cardinality if and only if there exists a bijective function between them:

$$|A| = |B| \quad \Leftrightarrow \quad \text{there exists bijective } f : A \to B.$$

A bijective function between two sets creates a correspondence between them.

8.4. **Sets of Functions.** Let A and B be sets. The set of all functions between then is

$$\mathcal{F}(A, B) = \{ \text{ functions } f : A \to B\}.$$

The set of all injective functions between them is

$$\mathcal{I}(A, B) = \{ \text{ functions } f : A \to B \mid \; f \text{ is injective } \}.$$

Suppose $|A| = k$ and $|B| = n$. Counting principles imply that

$$|\mathcal{F}(A, B)| = n^k \quad \text{and} \quad |\mathcal{I}(A, B)| = \frac{n!}{(n-k)!},$$

except when $k > n$, in which case $\mathcal{I}(A, B) = \varnothing$.

9. Exercises

Exercise 1. Let $A = \{4, 5, 6, 7, 8, 9, 10, 11\}$, $B = \{2, 4, 6, 8, 10, 12, 14, 16\}$, and $C = \{3, 6, 9, 12, 15, 18, 21\}$.
Find the indicated set.

- **(a)** $(A \cap B) \setminus C$
- **(b)** $A \setminus (B \cup C)$
- **(c)** $(A \setminus B) \cup C$

Exercise 2. Let $A = [0, 5]$, $B = (2, 7)$, $C = (6, 9)$, and $D = \{1, 3, 4, 7\}$. Find each of the following sets.

- **(a)** $(A \cup B) \setminus D$
- **(b)** $B \cup (C \cap D)$
- **(C)** $A \setminus D$
- **(D)** $(A \cup C) \setminus D$

Exercise 3. Let $A = \{x \in \mathbb{R} \mid -3 \le x < 7\}$ and $B = \{x \in \mathbb{R} \mid 1 < x \le 5\}$.
Find the indicated set.

- **(a)** A
- **(b)** B
- **(c)** $A \cup B$
- **(d)** $A \cap B$
- **(e)** $A \setminus B$

Exercise 4. Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{1, 3, 5, 7, 9, 11\}$.
Find $C = (A \cup B) \setminus (A \cap B)$.

Exercise 5. Let $D = [2, 10]$ and $E = (\pi, 8]$. Find $F = (D \setminus E) \setminus \mathbb{Z}$.

Exercise 6. Sketch the graph of the set $[1, 3] \times ([1, 4] \setminus [2, 3])$ as a subset of $\mathbb{R}^2$.

Exercise 7. Sketch the graph of the set

$$([1, 5] \setminus (2, 4)) \times (\{1, 3\} \cup [4, 5]).$$

Exercise 8. Let $A = [2, 3) \cup \{4\} \cup (5, 6]$. Sketch the graph of the set $A \times A$.

Exercise 9. Draw Venn diagrams which demonstrate the following equations.

- **(a)** $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- **(b)** $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- **(c)** $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$
- **(d)** $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

Exercise 10. Let A and B be subsets of a set U. The *symmetric difference* of A and B, denoted $A \triangle B$, is the set of points in U which are in either A or B but not in both.

- **(a)** Draw a Venn diagram describing $A \triangle B$.
- **(b)** Find two set expressions which could be used to define $A \triangle B$. These expressions may use A, B, union, intersection, complement, and parentheses,

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